

RULED MINIMAL SURFACES IN THE THREE DIMENSIONAL HEISENBERG GROUP

YOUNG WOOK KIM, SUNG-EUN KOH, HYUNG YONG LEE, HEAYONG SHIN,
AND SEONG-DEOG YANG

To the memory of Professor Seok Woo Kim

ABSTRACT. It is shown that parts of planes, helicoids and hyperbolic paraboloids are the only minimal surfaces ruled by geodesics in the three dimensional Riemannian Heisenberg group. It is also shown that they are the only surfaces in the three dimensional Heisenberg group whose mean curvature is zero with respect to both of the standard Riemannian metric and the standard Lorentzian metric.

1. INTRODUCTION

The three dimensional Heisenberg group \mathbb{H}_3 is the two-step nilpotent Lie group standardly represented in $GL_3(\mathbb{R})$ by

$$\begin{bmatrix} 1 & x & z + \frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}.$$

We consider in this paper two left invariant metrics on \mathbb{H}_3 , one is Riemannian and the other Lorentzian. Let us denote by Nil^3 the 3-dimensional Heisenberg group \mathbb{H}_3 endowed with the left-invariant Riemannian metric

$$g = dx^2 + dy^2 + \left(dz + \frac{1}{2}(ydx - xdy) \right)^2$$

on \mathbb{R}^3 . The Riemannian Heisenberg group Nil^3 is a three dimensional homogeneous manifold with a 4-dimensional isometry group; hence it is the most simple 3-manifold apart from the space-forms. Moreover, it is a Riemannian fibration over the Euclidean plane \mathbb{R}^2 , with the projection $(x, y, z) \mapsto (x, y)$.

In the first part of this paper, we give a classification of all ruled minimal surfaces in Nil^3 . In order for this, we first show in Lemma 2.1 that if a ruled surface is minimal and if a ruling geodesic is not tangent to the fibre, then the ruled surface should be horizontally ruled. That is, its ruling geodesics are orthogonal to the fibres. In fact, it was one of the key observations in classifying the ruled minimal surfaces in $\mathbb{S}^2 \times \mathbb{R}$ or in $\mathbb{H}^2 \times \mathbb{R}$ in our

Date: June 6, 2009.

2000 Mathematics Subject Classification. 53A35 .

Key words and phrases. Heisenberg group, ruled surface, minimal surface.

The second named author was supported by KRF-2008-313-C00069.

previous paper [10]. It turns out that this fact simplifies the nonlinear partial differential equations describing ruled minimal surfaces. Then we show in Theorem 2.3 that any ruled minimal surface in Nil^3 is, up to isometries, a part of the horizontal plane $z = 0$, the vertical plane $y = 0$, a helicoid $\tan(\lambda z) = \frac{y}{x}, \lambda \neq 0$ or a hyperbolic paraboloid $z = -\frac{xy}{2}$, see §2.3 for the definition of planes. Moreover, we show in §2.8 that all of them can be regarded as helicoids or the limits of the sequences of helicoids in the Gromov-Hausdorff sense.

In fact, it was shown in [4] that, up to isometries, parts of planes, the helicoids and the hyperbolic paraboloids are the only minimal surfaces in Nil^3 ruled by straight lines which are geodesics. According to Lemma 2.1, any ruling geodesic of a ruled minimal surface is either parallel or orthogonal to the fibres. We then note that geodesics parallel or orthogonal to the fibres everywhere are straight lines (in the Euclidean sense) in Lemma 2.4 and thereby show that “straight line” condition may be deleted in the aforementioned claim. For the properties of the Gauss map and representation formulae of the minimal surfaces in Nil^3 , see for example [3], [5], [7], [8], [13], [17].

In the second part, we consider the natural left invariant Lorentzian metric

$$g_L = dx^2 + dy^2 - \left(dz + \frac{1}{2}(ydx - xdy) \right)^2$$

on \mathbb{H}_3 . (Lorentzian metrics on \mathbb{H}_3 are discussed in [15], [16]). Then we consider surfaces in \mathbb{H}_3 whose mean curvature is zero with respect to both metrics g and g_L and show that they must be one of the above mentioned surfaces, that is, a part of planes, helicoids or hyperbolic paraboloids in Theorem 3.2. It can be considered as a generalization of the fact that the helicoids are the only surfaces except the planes in \mathbb{R}^3 whose mean curvature is zero with respect to both the standard Riemannian metric and the standard Lorentzian metric [12] and the fact that the helicoids (surfaces invariant under the screw motion) are the only surfaces except the trivial ones in $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$ whose mean curvature is zero with respect to both the standard Riemannian metric and the standard Lorentzian metric [10]. In order for this we derive the equation for the mean curvature of a graph in \mathbb{H}_3 to be zero with respect to the Lorentzian metric g_L and compare it with the minimal surface equation. We would like to remark that the idea of considering these two equations in the same time is not new, see also for example, [1], [2], [12].

2. RULED MINIMAL SURFACES IN Nil^3

We first state several facts on the geometry of Nil^3 , necessary for the proof of the main result in this section. For their proofs, one may refer, for example, to [9].

2.1. A Frame Field. It can be easily seen that

$$\mathbf{e}_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}, \quad \mathbf{e}_2 = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}, \quad \mathbf{e}_3 = \frac{\partial}{\partial z}$$

is a left invariant orthonormal frame field on Nil^3 and in particular, \mathbf{e}_3 is tangent to the fibres. Let ∇ be the Levi-Civita connection on Nil^3 , then, for this frame field we have,

$$\begin{aligned}\nabla_{\mathbf{e}_i}\mathbf{e}_i &= 0, \quad i = 1, 2, 3, \\ \nabla_{\mathbf{e}_1}\mathbf{e}_2 &= -\nabla_{\mathbf{e}_2}\mathbf{e}_1 = \frac{1}{2}\mathbf{e}_3, \quad \nabla_{\mathbf{e}_1}\mathbf{e}_3 = \nabla_{\mathbf{e}_3}\mathbf{e}_1 = -\frac{1}{2}\mathbf{e}_2, \quad \nabla_{\mathbf{e}_2}\mathbf{e}_3 = \nabla_{\mathbf{e}_3}\mathbf{e}_2 = \frac{1}{2}\mathbf{e}_1.\end{aligned}$$

2.2. Isometries. The isometry group of Nil^3 has two connected components: an isometry either preserves the orientation of both the fibres and the base of the fibration, or reverses both orientations. The identity component of the isometry group of Nil^3 is isomorphic to $SO(2) \ltimes \mathbb{R}^3$ whose action is given by

$$\begin{aligned}& \left(\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}, \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ \frac{1}{2}(a \sin \theta - b \cos \theta) & \frac{1}{2}(a \cos \theta + b \sin \theta) & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} a \\ b \\ c \end{bmatrix},\end{aligned}$$

which shows that Nil^3 is a homogeneous space. In fact, one can see that, for any point $p \in \mathbb{H}_3$ and a unit tangent vector \mathbf{v} orthogonal to $\mathbf{e}_3(p)$, there exists a unique isometry φ such that $\varphi(p) = \mathbf{0}$, $d\varphi(\mathbf{v}) = \mathbf{e}_1(\mathbf{0})$ and $d\varphi(\mathbf{e}_3(p)) = \mathbf{e}_3(\mathbf{0})$. Note also that the translations along the z axis (in the Euclidean sense) are isometries belonging to the identity component.

2.3. Euclidean Planes. A *Euclidean plane* or simply a *plane* is a set of points $(x, y, z) \in \mathbb{H}_3$ satisfying a linear equation $ax + by + cz + d = 0$. It is easy to see that all the planes except the “vertical” planes $ax + by + d = 0$ are congruent. In fact, every nonvertical plane $ax + by + z + d = 0$ is isometric to the “horizontal” plane $z = 0$ via, for example the isometry

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -a & -b & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} -2b \\ 2a \\ -d \end{bmatrix}.$$

Moreover, a vertical plane is not congruent to a nonvertical plane since every isometric image of a fibre is a fibre. In fact, one can check that a vertical plane is not isometric to a nonvertical plane by computing their curvatures.

2.4. A parametrization of ruled surfaces. Let Σ be a ruled surface in Nil^3 and let $p \in \Sigma$ be a point at which $T_p\Sigma$ is transversal to the fibre. Assume, furthermore, that the direction of the ruling geodesic at p is not perpendicular to the fibres. Then, in a neighborhood of p , we can take a tangent vector field V to Σ which is in the direction of the ruling everywhere on the neighborhood as

$$V = \eta(\cos \theta \mathbf{e}_1 - \sin \theta \mathbf{e}_2) + \mathbf{e}_3$$

for some functions η and θ on Σ . Since $T_p\Sigma$ is transversal to the fibre, the unit normal vector field \mathbf{n} of Σ is not perpendicular to \mathbf{e}_3 , $\langle \mathbf{n}, \mathbf{e}_3 \rangle \neq 0$. Then

$$W = \sin \theta \mathbf{e}_1 + \cos \theta \mathbf{e}_2 - \frac{\langle \mathbf{n}, \sin \theta \mathbf{e}_1 + \cos \theta \mathbf{e}_2 \rangle}{\langle \mathbf{n}, \mathbf{e}_3 \rangle} \mathbf{e}_3$$

gives another tangent vector field on Σ which is transversal to V . Now we take a parametrization $X(s, t)$ of Σ in the neighborhood of p such that $X(s, 0)$ is the integral curve of W with $X(0, 0) = p$ and such that t parameter curves are the ruling geodesics with $X_t(s, 0) = V(X(s, 0))$. Then $X(s, t)$ is a parametrization of the ruled surface Σ in the neighborhood of p satisfying

$$(1) \quad \begin{aligned} X_s(s, 0) &= \sin \alpha(s) \mathbf{e}_1 + \cos \alpha(s) \mathbf{e}_2 + g(s) \mathbf{e}_3, \\ X_t(s, 0) &= h(s)(\cos \alpha(s) \mathbf{e}_1 - \sin \alpha(s) \mathbf{e}_2) + \mathbf{e}_3, \\ \nabla_{X_t} X_t &= 0 \end{aligned}$$

for some smooth functions $h(s)$, $\alpha(s)$ and $g(s)$.

For the parametrization X satisfying the condition (1), we are to compute the functions X_{si} and X_{ti} defined by

$$\begin{aligned} X_s(s, t) &= X_{s1}(s, t) \mathbf{e}_1 + X_{s2}(s, t) \mathbf{e}_2 + X_{s3}(s, t) \mathbf{e}_3, \\ X_t(s, t) &= X_{t1}(s, t) \mathbf{e}_1 + X_{t2}(s, t) \mathbf{e}_2 + X_{t3}(s, t) \mathbf{e}_3. \end{aligned}$$

Now, since t parameter curves are geodesics, we have

$$\begin{aligned} \nabla_{X_t} X_t &= \sum_i \frac{\partial X_{ti}}{\partial t} \mathbf{e}_i + \sum_{i,j} X_{ti} X_{tj} \nabla_{\mathbf{e}_i} \mathbf{e}_j \\ &= \left(\frac{\partial X_{t1}}{\partial t} + X_{t2} X_{t3} \right) \mathbf{e}_1 + \left(\frac{\partial X_{t2}}{\partial t} - X_{t1} X_{t3} \right) \mathbf{e}_2 + \frac{\partial X_{t3}}{\partial t} \mathbf{e}_3 = 0. \end{aligned}$$

By solving the system of equations

$$\frac{\partial X_{t1}}{\partial t} + X_{t2} X_{t3} = 0, \quad \frac{\partial X_{t2}}{\partial t} - X_{t1} X_{t3} = 0, \quad \frac{\partial X_{t3}}{\partial t} = 0$$

with the initial condition

$$X_{t1}(s, 0) = h(s) \cos \alpha(s), \quad X_{t2}(s, 0) = -h(s) \sin \alpha(s), \quad X_{t3}(s, 0) = 1$$

we have

$$X_{t1}(s, t) = h(s) \cos(t - \alpha(s)), \quad X_{t2}(s, t) = h(s) \sin(t - \alpha(s)), \quad X_{t3}(s, t) = 1.$$

On the other hand, since the Levi-Civita connection ∇ is torsion free, one has

$$\nabla_{X_t} X_s = \nabla_{X_s} X_t.$$

Hence we have

$$\begin{aligned}
& \left(\frac{\partial X_{s1}}{\partial t} + \frac{1}{2}(X_{t2}X_{s3} + X_{t3}X_{s2}) \right) \mathbf{e}_1 + \left(\frac{\partial X_{s2}}{\partial t} - \frac{1}{2}(X_{t1}X_{s3} + X_{t3}X_{s1}) \right) \mathbf{e}_2 \\
& \quad + \left(\frac{\partial X_{s3}}{\partial t} + \frac{1}{2}(X_{t1}X_{s2} - X_{t2}X_{s1}) \right) \mathbf{e}_3 \\
&= \left(\frac{\partial X_{t1}}{\partial s} + \frac{1}{2}(X_{s2}X_{t3} + X_{s3}X_{t2}) \right) \mathbf{e}_1 + \left(\frac{\partial X_{t2}}{\partial s} - \frac{1}{2}(X_{s1}X_{t3} + X_{s3}X_{t1}) \right) \mathbf{e}_2 \\
& \quad + \left(\frac{\partial X_{t3}}{\partial s} + \frac{1}{2}(X_{s1}X_{t2} - X_{s2}X_{t1}) \right) \mathbf{e}_3,
\end{aligned}$$

and X_{si} satisfies the equations

$$\begin{aligned}
\frac{\partial X_{s1}}{\partial t} &= \frac{\partial X_{t1}}{\partial s} = h'(s) \cos(t - \alpha(s)) + h(s) \alpha'(s) \sin(t - \alpha(s)), \\
\frac{\partial X_{s2}}{\partial t} &= \frac{\partial X_{t2}}{\partial s} = h'(s) \sin(t - \alpha(s)) - h(s) \alpha'(s) \cos(t - \alpha(s)), \\
\frac{\partial X_{s3}}{\partial t} &= \frac{\partial X_{t3}}{\partial s} + (X_{s1}X_{t2} - X_{s2}X_{t1}) \\
&= h(s) \sin(t - \alpha(s))X_{s1} - h(s) \cos(t - \alpha(s))X_{s2}
\end{aligned}$$

with the initial condition

$$X_{s1}(s, 0) = \sin \alpha(s), X_{s2}(s, 0) = \cos \alpha(s), X_{t3}(s, 0) = g(s).$$

By solving these equations, we get

$$\begin{aligned}
X_{s1}(s, t) &= \sin \alpha(s) + h'(s) \sin(t - \alpha(s)) + h'(s) \sin \alpha(s) \\
&\quad - h(s) \alpha'(s) \cos(t - \alpha(s)) + h(s) \alpha'(s) \cos \alpha(s), \\
X_{s2}(s, t) &= \cos \alpha(s) - h'(s) \cos(t - \alpha(s)) + h'(s) \cos \alpha(s) \\
&\quad - h(s) \alpha'(s) \sin(t - \alpha(s)) - h(s) \alpha'(s) \sin \alpha(s), \\
X_{s3}(s, t) &= g(s) - h(s) \sin t + th(s)h'(s) - h(s)h'(s) \sin t \\
&\quad + h(s)^2 \alpha'(s) - h(s)^2 \alpha'(s) \cos t.
\end{aligned}$$

2.5. The second derivatives of X . We are to compute the derivatives $\nabla_{X_t} X_t, \nabla_{X_s} X_t = \nabla_{X_t} X_s$ and $\nabla_{X_s} X_s$. For notational simplicity, let us set

$$\begin{aligned}
X_{t;t} &:= \nabla_{X_t} X_t = X_{tt1} \mathbf{e}_1 + X_{tt2} \mathbf{e}_2 + X_{tt3} \mathbf{e}_3, \\
X_{s;t} &:= \nabla_{X_t} X_s = X_{st1} \mathbf{e}_1 + X_{st2} \mathbf{e}_2 + X_{st3} \mathbf{e}_3, \\
X_{s;s} &:= \nabla_{X_s} X_s = X_{ss1} \mathbf{e}_1 + X_{ss2} \mathbf{e}_2 + X_{ss3} \mathbf{e}_3.
\end{aligned}$$

Since t parameter curves are geodesics, we have $X_{t;t} = 0$, that is,

$$X_{tt1} = X_{tt2} = X_{tt3} = 0.$$

From

$$\begin{aligned}
X_{s;t} &= X_{t;s} \\
&= \left(\frac{\partial X_{s1}}{\partial t} + \frac{1}{2}(X_{t2}X_{s3} + X_{t3}X_{s2}) \right) \mathbf{e}_1 + \left(\frac{\partial X_{s2}}{\partial t} - \frac{1}{2}(X_{t1}X_{s3} + X_{t3}X_{s1}) \right) \mathbf{e}_2 \\
&\quad + \left(\frac{\partial X_{s3}}{\partial t} + \frac{1}{2}(X_{t1}X_{s2} - X_{t2}X_{s1}) \right) \mathbf{e}_3, \\
X_{s;s} &= \left(\frac{\partial X_{s1}}{\partial s} + X_{s2}X_{s3} \right) \mathbf{e}_1 + \left(\frac{\partial X_{s2}}{\partial s} - X_{s1}X_{s3} \right) \mathbf{e}_2 + \frac{\partial X_{s3}}{\partial s} \mathbf{e}_3
\end{aligned}$$

we have

$$\begin{aligned}
X_{st1} &= \frac{1}{2} \left[\cos \alpha(s) + h'(s) \cos(t - \alpha(s)) + h'(s) \cos \alpha(s) \right. \\
&\quad \left. + h(s) \alpha'(s) \sin(t - \alpha(s)) - h(s) \alpha'(s) \sin \alpha(s) \right. \\
&\quad \left. + h(s) \sin(t - \alpha(s)) \left(g(s) + h(s) \left(-\sin t + h'(s)(t - \sin t) + 2h(s) \alpha'(s) \sin^2 t/2 \right) \right) \right], \\
X_{st2} &= \frac{1}{2} \left[-\sin \alpha(s) + h'(s) \sin(t - \alpha(s)) - h'(s) \sin \alpha(s) \right. \\
&\quad \left. - h(s) \alpha'(s) \cos(t - \alpha(s)) - h(s) \alpha'(s) \cos \alpha(s) \right. \\
&\quad \left. + h(s) \cos(t - \alpha(s)) \left(-g(s) + h(s) \left(\sin t - h'(s)(t - \sin t) - 2h(s) \alpha'(s) \sin^2 t/2 \right) \right) \right], \\
X_{st3} &= \frac{1}{2} h(s) \left[-\cos t - h'(s)(\cos t - 1) + h(s) \alpha'(s) \sin t \right], \\
X_{ss1} &= \alpha'(s) \cos \alpha(s) - 2h'(s) \alpha'(s) \cos(t - \alpha(s)) + 2h'(s) \alpha'(s) \cos \alpha(s) \\
&\quad - h(s) \alpha'(s)^2 \sin(t - \alpha(s)) - h(s) \alpha'(s)^2 \sin \alpha(s) \\
&\quad + \left(-\cos \alpha(s) + \cos(t - \alpha(s)) - h'(s) \cos \alpha(s) + h(s) \sin(t - \alpha(s)) + \alpha'(s) \sin \alpha(s) \right) \\
&\quad \left(-g(s) + h(s) (\sin t + h'(s)(\sin t - t) - 2h(s) \alpha'(s) \sin^2 t/2) \right) \\
&\quad + h''(s) \sin(t - \alpha(s)) + h''(s) \sin \alpha(s) - h(s) \alpha''(s) \cos(t - \alpha(s)) + h(s) \alpha''(s) \cos \alpha(s), \\
X_{ss2} &= -\alpha'(s) \sin \alpha(s) - 2h'(s) \alpha'(s) \sin(t - \alpha(s)) - 2h'(s) \alpha'(s) \sin \alpha(s) \\
&\quad + h(s) \alpha'(s)^2 \cos(t - \alpha(s)) - h(s) \alpha'(s)^2 \cos \alpha(s) \\
&\quad + \left(\sin \alpha(s) + h'(s) (\sin(t - \alpha(s)) + \sin \alpha(s)) + 2h(s) \alpha'(s) \sin t/2 \sin(t/2 - \alpha(s)) \right) \\
&\quad \left(-g(s) + h(s) (\sin t + h'(s)(\sin t - t) - 2h(s) \alpha'(s) \sin^2 t/2) \right) \\
&\quad - h''(s) \cos(t - \alpha(s)) + h''(s) \cos \alpha(s) - h(s) \alpha''(s) \sin(t - \alpha(s)) - h(s) \alpha''(s) \sin \alpha(s), \\
X_{ss3} &= g'(s) + h'(s)^2(t - \sin t) - h'(s) (\sin t - 4h(s) \alpha'(s) \sin^2 t/2) \\
&\quad + h(s) (h''(s)(t - \sin t) - h(s) \alpha''(s)(\cos t - 1)).
\end{aligned}$$

2.6. Mean curvature. We give a condition for the ruled surface Σ to be minimal in terms of the parametrization X . Now let E, F, G be the coefficients of the first fundamental form and l, m, n those of the second fundamental form of the surface Σ whose parametrization satisfies (1). Then the mean curvature of Σ in a neighborhood of p is given by

$$\begin{aligned} H &= \frac{1}{2} \frac{Gl - 2Fm + En}{EG - F^2} \\ &= \frac{1}{2} \frac{\langle X_t, X_t \rangle \langle X_{s;s}, X_s \times X_t \rangle - 2\langle X_s, X_t \rangle \langle X_{s;t}, X_s \times X_t \rangle}{\|X_s \times X_t\|^3}. \end{aligned}$$

Since

$$X_s \times X_t = (X_{s2}X_{t3} - X_{s3}X_{t2})\mathbf{e}_1 + (X_{s3}X_{t1} - X_{s1}X_{t3})\mathbf{e}_2 + (X_{s1}X_{t2} - X_{s2}X_{t1})\mathbf{e}_3,$$

X is a parametrization of a minimal surface if and only if

$$\begin{aligned} \tilde{H} &:= \langle X_t, X_t \rangle \langle X_{s;s}, X_s \times X_t \rangle - 2\langle X_s, X_t \rangle \langle X_{s;t}, X_s \times X_t \rangle \\ &= \left(\sum_i X_{ti}^2 \right) \left((X_{s2}X_{t3} - X_{s3}X_{t2})X_{ss1} \right. \\ &\quad \left. + (X_{s3}X_{t1} - X_{s1}X_{t3})X_{ss2} + (X_{s1}X_{t2} - X_{s2}X_{t1})X_{ss3} \right) \\ (2) \quad &- 2 \left(\sum_i X_{si}X_{ti} \right) \left((X_{s2}X_{t3} - X_{s3}X_{t2})X_{st1} \right. \\ &\quad \left. + (X_{s3}X_{t1} - X_{s1}X_{t3})X_{st2} + (X_{s1}X_{t2} - X_{s2}X_{t1})X_{st3} \right) \\ &= 0 \end{aligned}$$

2.7. Ruled minimal surfaces in Nil^3 . Now we are to find all ruled minimal surfaces in Nil^3 .

Lemma 2.1. *If the surface whose parametrization X satisfies (1) is minimal, then $h(s) = 0$ for all s .*

[Proof] Considering the parametrizations $\tilde{X}(s, t) := X(s - s_0, t)$ if necessary, we need only to prove $h(0) = 0$. By rotating the surface in Nil^3 if necessary, we may assume that $\alpha(0) = 0$. Since we have explicit formulae for all $X_s, X_t, X_{s;s}, X_{s;t}, X_{t;t}$, we can compute \tilde{H} directly. In particular, since X is minimal, we have $\tilde{H}(0, t) = 0$ for all t . Since $\alpha(0) = 0$, $\tilde{H}(0, t)$ becomes

$$\begin{aligned} \tilde{H}(0, t) &= A_0 + A_1t + A_2t^2 + A_3t^3 \\ &\quad + B_0 \cos t + B_1t \cos t + B_2t^2 \cos t + B_3 \cos 2t + B_4t \cos 2t + B_5 \cos 3t \\ &\quad + C_0 \sin t + C_1t \sin t + C_2t^2 \sin t + C_3 \sin 2t + C_4t \sin 2t + C_5 \sin 3t \end{aligned}$$

where the constants A_i, B_i, C_i are functions of $h(0), h'(0), h''(0), \alpha'(0), \alpha''(0)$ and $g(0), g'(0)$. In the following computation, we are to use only the following terms:

$$A_3 = h(0)^5 h'(0)^3,$$

$$\begin{aligned}
B_1 &= -3h(0)h'(0)^2 - h(0)^3h'(0)^2 - 3h(0)h'(0)^3 - h(0)^3h'(0)^3 - 2h(0)^3g(0)h'(0)\alpha'(0) \\
&\quad - 6g(0)h(0)^5h'(0)\alpha'(0) - 3h(0)^3h'(0)\alpha'(0)^2 - 9h'(0)h(0)^5\alpha'(0)^2 - 6h(0)^7h'(0)\alpha'(0)^2 \\
&\quad - h(0)^4h''(0) - h(0)^2h''(0), \\
B_5 &= \frac{1}{4} \left(3h(0)^4\alpha'(0) + 3h(0)^6\alpha'(0) + 6h(0)^4h'(0)\alpha'(0) + 6h(0)^6h'(0)\alpha'(0) \right. \\
&\quad \left. + 3h(0)^4h'(0)^2\alpha'(0) + 3h'(0)^2\alpha'(0)h(0)^6 - h(0)^6\alpha'(0)^3 - h(0)^8\alpha'(0)^3 \right), \\
C_5 &= \frac{1}{4} \left(h(0)^3 + h(0)^5 + 3h(0)^3h'(0) + 3h(0)^5h'(0) + 3h(0)^3h'(0)^2 + 3h(0)^5h'(0)^2 \right. \\
&\quad \left. + h(0)^3h'(0)^3 + h(0)^5h'(0)^3 - 3h(0)^5\alpha'(0)^2 - 3h(0)^7\alpha'(0)^2 - 3h(0)^5h'(0)\alpha'(0)^2 \right. \\
&\quad \left. - 3h'(0)h(0)^7\alpha'(0)^2 \right).
\end{aligned}$$

Since $\tilde{H}(0, t) = 0$ for all t and since the above expression is a linear combination of linearly independent functions of t , all of A_i , B_i , C_i must be 0. Now from $A_3 = h(0)^5h'(0)^3 = 0$, we have either $h(0) = 0$ or $h'(0) = 0$. Now suppose $h(0) \neq 0$. Then $h'(0) = 0$ and B_1 becomes

$$B_1 = -h''(0)h(0)^4 - h''(0)h(0)^2 = -h''(0)h(0)^2(h(0)^2 + 1) = 0.$$

Hence we have $h''(0) = 0$ and in addition

$$4B_5 = -\alpha'(0)^3h(0)^8 - \alpha'(0)^3h(0)^6 + 3\alpha'(0)h(0)^6 + 3\alpha'(0)h(0)^4 = 0$$

$$4C_5 = -3\alpha'(0)^2h(0)^7 - 3\alpha'(0)^2h(0)^5 + h(0)^5 + h(0)^3 = 0.$$

Then, since

$$3B_5 - h(0)\alpha'(0)C_5 = 2\alpha'(0)h(0)^4(h(0)^2 + 1) = 0,$$

we have $\alpha'(0) = 0$ and C_5 becomes

$$4C_5 = h(0)^3(h(0)^2 + 1) = 0.$$

This contradicts the assumption $h(0) \neq 0$. Hence we must have $h(0) = 0$ if X is a parametrization of a minimal surface. \square

If p is a point in a ruled surface Σ at which $T_p\Sigma$ is transversal to the fibre and the direction of the ruling is not perpendicular to the fibres, then Σ has the parametrization of the type given in (1) in a neighborhood of p . If, in addition, Σ is minimal then the above lemma implies that the direction of the ruling at p is parallel to the fibres. This contradicts the fact that $T_p\Sigma$ is transversal to the fibres. Therefore we can conclude that in a ruled minimal surface Σ the directions of the rulings are horizontal, that is, perpendicular to the fibres wherever $T_p\Sigma$ is transversal to the fibres.

Now we consider the minimal surfaces which are ruled by horizontal geodesics.

Lemma 2.2. *If Σ is a minimal surface in Nil^3 ruled by geodesics perpendicular to the fibres, then up to the isometries in Nil^3 , Σ is a part of the horizontal plane $z = 0$, the vertical plane $y = 0$, a helicoid $\tan(\lambda z) = \frac{y}{x}, \lambda \neq 0$ or a hyperbolic paraboloid $z = -\frac{xy}{2}$.*

[Proof] One can see that the surface Σ has a local parametrization $Y(s, t)$ satisfying

$$(3) \quad \begin{aligned} Y_s(s, 0) &= \cos \beta(s)(-\sin \alpha(s)\mathbf{e}_1 + \cos \alpha(s)\mathbf{e}_2) + \sin \beta(s)\mathbf{e}_3, \\ Y_t(s, 0) &= \cos \alpha(s)\mathbf{e}_1 + \sin \alpha(s)\mathbf{e}_2, \\ \nabla_{Y_t} Y_t &= 0. \end{aligned}$$

If we set

$$\begin{aligned} Y_s(s, t) &= Y_{s1}(s, t)\mathbf{e}_1 + Y_{s2}(s, t)\mathbf{e}_2 + Y_{s3}(s, t)\mathbf{e}_3, \\ Y_t(s, t) &= Y_{t1}(s, t)\mathbf{e}_1 + Y_{t2}(s, t)\mathbf{e}_2 + Y_{t3}(s, t)\mathbf{e}_3, \end{aligned}$$

by solving the equation $\nabla_{Y_t} Y_t = 0$ with the initial condition

$$Y_t(s, 0) = \cos \alpha(s)\mathbf{e}_1 + \sin \alpha(s)\mathbf{e}_2$$

we have

$$Y_{t1}(s, t) = \cos \alpha(s), \quad Y_{t2}(s, t) = \sin \alpha(s), \quad Y_{t3}(s, t) = 0.$$

Moreover, from $\nabla_{Y_t} Y_s = \nabla_{Y_s} Y_t$, we can see that Y_{si} satisfies the equations

$$\begin{aligned} \frac{\partial Y_{s1}}{\partial t} &= \frac{\partial Y_{t1}}{\partial s} = -\alpha'(s) \sin \alpha(s), \\ \frac{\partial Y_{s2}}{\partial t} &= \frac{\partial Y_{t2}}{\partial s} = \alpha'(s) \cos \alpha(s), \\ \frac{\partial Y_{s3}}{\partial t} &= \frac{\partial Y_{t3}}{\partial s} + (Y_{s1}Y_{t2} - Y_{s2}Y_{t1}) = \sin \alpha(s)Y_{s1} - \cos \alpha(s)Y_{s2} \end{aligned}$$

with the initial condition

$$Y_{s1}(s, 0) = -\cos \beta(s) \sin \alpha(s), \quad Y_{s2}(s, 0) = \cos \beta(s) \cos \alpha(s), \quad Y_{s3}(s, 0) = \sin \beta(s).$$

By solving this system of equations, we get

$$\begin{aligned} Y_{s1}(s, t) &= -\cos \beta(s) \sin \alpha(s) - t\alpha'(s) \sin \alpha(s), \\ Y_{s2}(s, t) &= \cos \beta(s) \cos \alpha(s) + t\alpha'(s) \cos \alpha(s), \\ Y_{s3}(s, t) &= \sin \beta(s) - t \cos \beta(s) - \frac{1}{2}t^2\alpha'(s). \end{aligned}$$

By direct computations, we can see that the minimal surface equation (2) can be written as

$$\beta'(s) + t(\alpha'(s)\beta'(s) \cos \beta(s) - \alpha''(s) \sin \beta(s)) + \frac{t^2}{2}(\alpha'(s)\beta'(s) \sin \beta(s) + \alpha''(s) \cos \beta(s)) = 0.$$

Therefore we have $\beta'(s) = 0$ and $\alpha''(s) = 0$, that is, $\beta(s) = b$ and $\alpha(s) = as + c$ for some constants a, b, c .

When $a \neq 0$, relocating the surface Σ by an isometry in Nil^3 , we may assume that

$$\alpha(s) = as \quad \text{and} \quad Y(0, 0) = \left(\frac{\cos b}{a}, 0, 0 \right).$$

Then, since $\mathbf{e}_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}$, $\mathbf{e}_2 = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}$, $\mathbf{e}_3 = \frac{\partial}{\partial z}$, we have

$$\begin{aligned} Y_s(s, 0) &= -\cos b \sin(as) \mathbf{e}_1 + \cos b \cos(as) \mathbf{e}_2 + \sin b \mathbf{e}_3 \\ &= -\cos b \sin(as) \frac{\partial}{\partial x} + \cos b \cos(as) \frac{\partial}{\partial y} \\ &\quad + \left(\sin b + \frac{y}{2} \cos b \sin(as) + \frac{x}{2} \cos b \cos(as) \right) \frac{\partial}{\partial z}, \\ Y_t(s, t) &= \cos(as) \mathbf{e}_1 + \sin(as) \mathbf{e}_2 \\ &= \cos(as) \frac{\partial}{\partial x} + \sin(as) \frac{\partial}{\partial y} + \left(-\frac{y}{2} \cos(as) + \frac{x}{2} \sin(as) \right) \frac{\partial}{\partial z}. \end{aligned}$$

Integrating the components of $Y_s(s, 0)$ with initial data $Y(0, 0) = (\frac{\cos b}{a}, 0, 0)$, we have

$$Y(s, 0) = \left(\frac{1}{a} \cos b \cos(as), \frac{1}{a} \cos b \sin(as), \frac{s}{4a} (1 + \cos(2b) + 4a \sin b) \right).$$

Then integrating the components of $Y_t(s, t)$ with initial data $Y(s, 0)$, we have

$$Y(s, t) = \left(t \cos(as) + \frac{1}{a} \cos b \cos(as), t \sin(as) + \frac{1}{a} \cos b \sin(as), \frac{s}{4a} (1 + \cos(2b) + 4a \sin b) \right).$$

Noting that

$$Y(s, t) = \left(t \cos(as), t \sin(as), \frac{s}{4a} (1 + \cos(2b) + 4a \sin b) \right),$$

we can see that Y is a parametrization of either the helicoid

$$\tan \lambda z = \frac{y}{x} \quad \text{where} \quad \lambda = \frac{4a^2}{1 + \cos(2b) + 4a \sin b}$$

if $1 + \cos(2b) + 4a \sin b \neq 0$, or the plane $z = 0$ if $1 + \cos(2b) + 4a \sin b = 0$.

When $a = 0$ and $\cos b \neq 0$, we may assume up to isometries that $\alpha(s) = 0$ and $Y(0, 0) = (-\tan b, 0, 0)$. Then

$$\begin{aligned} Y_s(s, 0) &= \cos b \mathbf{e}_2 + \sin b \mathbf{e}_3 = \cos b \frac{\partial}{\partial y} + \left(\sin b + \frac{x}{2} \cos b \right) \frac{\partial}{\partial z}, \\ Y_t(s, t) &= \mathbf{e}_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}, \end{aligned}$$

and a similar computation as above gives

$$Y(s, t) = \left(t - \tan b, s \cos b, -\frac{1}{2} st \cos b + \frac{1}{2} s \sin b \right)$$

which is a parametrization of the hyperbolic paraboloid $z = -\frac{xy}{2}$. When $a = 0$ and $\cos b = 0$, we have $Y_s(s, 0) = \mathbf{e}_3$, $Y_t(s, t) = \mathbf{e}_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}$ and $Y(s, t)$ is a parametrization of the xz -plane if we set $Y(0, 0) = (0, 0, 0)$. \square

Theorem 2.3. *If Σ is a minimal surface in Nil^3 ruled by geodesics, then up to the isometries in Nil^3 , Σ is a part of the horizontal plane $z = 0$, the vertical plane $y = 0$, a helicoid $\tan(\lambda z) = \frac{y}{x}$, $\lambda \neq 0$ or a hyperbolic paraboloid $z = -\frac{xy}{2}$.*

[Proof] If there is a point $p \in \Sigma$ at which $T_p\Sigma$ is transversal to the fibres, then Σ is transversal to the fibres in a neighborhood of p . Therefore, from the argument following the Lemma 2.1, the ruling geodesics through any points in the neighborhood must be horizontal. Then by the Lemma 2.2 the neighborhood coincides with a part of the helicoids, the hyperbolic paraboloid or the xy -plane up to the isometries in Nil^3 . Now since the tangent spaces at every points of these surfaces are transversal to fibres, the whole Σ must be a part of one of these surfaces.

On the other hand, if the tangent space $T_p\Sigma$ is tangent to the fibres at every point $p \in \Sigma$, then \mathbf{e}_3 is tangent to Σ . Relocating Σ by an isometry of Nil^3 , we may assume that $(0,0,0) \in \Sigma$ and that Σ is tangent to the plane $y = 0$ at $(0,0,0)$. So Σ is ruled by the fibres and has a ruled parametrization $X(s, t) = (x(s), y(s), t)$ satisfying $x(0) = y(0) = 0$, $y'(0) = 0$ and $x'(0) = 1$. The mean curvature of this parametrized surface can be easily computed to be

$$\frac{x''(s)y'(s) - x'(s)y''(s)}{(x'(s)^2 + y'(s)^2)^{3/2}}.$$

Solving the equation $x''(s)y'(s) - x'(s)y''(s) = 0$ with the above initial conditions, we have $y(s) = 0$ which implies that Σ is a part of the vertical plane $y = 0$. \square

By the above theorem, we know that the ruled minimal surfaces in Nil^3 are congruent to the surfaces given in the theorem which are all ruled by horizontal geodesics. In fact, the vertical plane $y = 0$ is also ruled by vertical geodesics, i.e., fibres and this is the only doubly ruled surface among the surfaces in Theorem 2.3. Noting that isometries in Nil^3 always moves fibres to fibres, we can see that the ruled minimal surfaces in Nil^3 always have horizontal ruling geodesics.

2.8. Ruled minimal surfaces as a limit of helicoids. Consider the (generic) helicoids

$$H_\lambda : y - x \tan(\lambda z) = 0$$

and the point $p_\lambda(r_\lambda, 0, 0)$ on the x -axis, where $r_\lambda = \sqrt{2/\lambda}$. The isometry which sends x -axis to itself and sends the origin to p_λ is given by the formula

$$(x, y, z) \mapsto \left(x + r_\lambda, y, z + \frac{r_\lambda}{2}y\right).$$

If we pull back H_λ via this isometry, then p_λ is moved to the origin and the equation of the pullback of H_λ becomes

$$y - (x + r_\lambda) \tan\left(\lambda z + \frac{r_\lambda \lambda}{2}y\right) = 0.$$

Now we multiply this equation by r_λ then a simple computation shows that the equation is of the form

$$z + \frac{xy}{2} + O(\sqrt{\lambda}) = 0$$

and as $\lambda \rightarrow 0$ this converges to the equation of the ruled minimal surface given by

$$z + \frac{xy}{2} = 0.$$

This shows that the pointed helicoids (H_λ, p_λ) converge (in the Gromov-Hausdorff sense [6]) to the exceptional ruled minimal surface $z + xy/2 = 0$:

$$(H_\lambda, p_\lambda) \rightarrow \{z + xy/2 = 0\} \quad \text{as } \lambda \rightarrow 0 + .$$

On the other hand, one can easily check that

$$(H_\lambda, 0) \rightarrow \text{horizontal plane} \quad \text{as } \lambda \rightarrow \infty,$$

$$(H_\lambda, 0) \rightarrow \text{vertical plane} \quad \text{as } \lambda \rightarrow 0.$$

Therefore all the ruled minimal surface in Nil^3 are either the helicoids or the limits of sequences of them.

2.9. Straight Line Geodesics. We characterize the geodesics which are straight lines in the Euclidean sense and give another proof of the result in [4] mentioned in the Introduction.

Proposition 2.4. *Let $\gamma(t) = (x(t), y(t), z(t))$ be a geodesic in Nil^3 .*

- (1) *If $\gamma'(0)$ is perpendicular to the fibre, then $\gamma(t)$ is a straight line everywhere perpendicular to the fibres.*
- (2) *If $\gamma'(0)$ is parallel to the fibre, then $\gamma(t)$ is a straight line everywhere parallel to the fibres.*

[Proof] Note first that

$$\gamma' = x' \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + z' \frac{\partial}{\partial z} = x' \mathbf{e}_1 + y' \mathbf{e}_2 + \left(z' + \frac{1}{2}(x'y - xy') \right) \mathbf{e}_3.$$

Then we have

$$\begin{aligned} \nabla_{\gamma'} \gamma' &= x'' \mathbf{e}_1 + y'' \mathbf{e}_2 + \left(z' + \frac{1}{2}(x'y - xy') \right)' \mathbf{e}_3 \\ &\quad + x' \nabla_{\gamma'} \mathbf{e}_1 + y' \nabla_{\gamma'} \mathbf{e}_2 + \left(z' + \frac{1}{2}(x'y - xy') \right) \nabla_{\gamma'} \mathbf{e}_3 \\ &= \left(x'' + y'(z' + \frac{1}{2}(x'y - xy')) \right) \mathbf{e}_1 + \left(y'' - x'(z' + \frac{1}{2}(x'y - xy')) \right) \mathbf{e}_2 \\ &\quad + \left(z' + \frac{1}{2}(x'y - xy') \right)' \mathbf{e}_3. \end{aligned}$$

Hence $\gamma(t) = (x(t), y(t), z(t))$ is a geodesic if and only if

$$\begin{aligned} (4) \quad &x'' + y'(z' + \frac{1}{2}(x'y - xy')) = 0, \\ &y'' - x'(z' + \frac{1}{2}(x'y - xy')) = 0, \\ &\left(z' + \frac{1}{2}(x'y - xy') \right)' = 0. \end{aligned}$$

Note that the straight line $(a, b, ct + d)$ parallel to the fibre is a geodesic. Now, suppose $\langle \gamma'(0), \mathbf{e}_3 \rangle = 0$. Then, since

$$\langle \gamma'(0), \mathbf{e}_3 \rangle = \left(z' + \frac{1}{2}(x'y - xy') \right) (0) = 0$$

and since $z' + \frac{1}{2}(x'y - xy')$ is a constant function from the geodesic equation (4), we have $z' + \frac{1}{2}(x'y - xy') = 0$ for all t . Moreover, the geodesic equation (4) gives

$$x''(t) = y''(t) = 0,$$

that is, $x(t)$ and $y(t)$ is a linear function of t and consequently from the geodesic equation (4) again, we have

$$z(t) = -\frac{1}{2}(x'(0)y(0) - x(0)y'(0))t + c$$

for a constant c . Now it is easy to see that $\gamma(t)$ is perpendicular to the fibres everywhere.

If $\gamma'(0)$ is parallel to the fibre, then the fibre through $\gamma(0)$ is an image of a geodesic, from the uniqueness of the geodesic, we have $\gamma(t) = (x(0), y(0), at + b)$ for constants a, b which is parallel to the fibre everywhere. \square

Proposition 2.5. *Suppose the straight line $\delta(t) = (a_1t + b_1, a_2t + b_2, a_3t + b_3)$ is a geodesic in Nil^3 . Then $\delta'(0) = (a_1, a_2, a_3)$ is either perpendicular or parallel to the fibre. Moreover, if $\delta'(0)$ is perpendicular to the fibre, then $\delta(t)$ is perpendicular to the fibre everywhere and if $\delta'(0)$ is parallel to the fibre, then $\delta(t)$ is parallel to the fibre everywhere.*

[Proof] In the proof of the above Proposition 2.4, one can see that in order for the straight line $\delta(t)$ to be a geodesic, it should be that

$$a_3 = -\frac{1}{2}(a_1b_2 - a_2b_1).$$

The claims follow easily from this fact. \square

Now we can also say that every ruled minimal surfaces in Nil^3 is ruled by geodesics which are also straight lines. We remark that it was shown in [4] that if the surface is ruled by geodesics which are also straight lines then the surface must be a part of the planes, helicoids or hyperbolic paraboloids, however, in view of Theorem 2.3, we can see that the “straight line” condition is redundant. On the other hand, one may get Theorem 2.3 by applying the aforementioned result together with Lemma 2.1 and Proposition 2.4.

3. ANOTHER CHARACTERIZATION OF RULED MINIMAL SURFACES IN \mathbb{H}_3 .

We consider surfaces in \mathbb{H}_3 whose mean curvature is zero with respect to both metrics g and g_L and show that they must be one of (a part of) the above mentioned surfaces, that is, planes, helicoids and hyperbolic paraboloids.

3.1. A Lorentzian connection. Let us consider the left-invariant Lorentzian metric

$$g_L = dx^2 + dy^2 - \left(dz + \frac{1}{2}(ydx - xdy) \right)^2$$

on \mathbb{H}_3 and let $\langle \cdot, \cdot \rangle$ be the Lorentzian inner product. Let $\mathbf{e}_1, \mathbf{e}_2$ and \mathbf{e}_3 be the same as the ones given in §2. It is easy to show that they are orthonormal with respect to the Lorentzian metric g_L as well, that is, $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0$ if $i \neq j$ and

$$\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = \langle \mathbf{e}_2, \mathbf{e}_2 \rangle = 1, \quad \langle \mathbf{e}_3, \mathbf{e}_3 \rangle = -1.$$

Now let D be the Levi-Civita connection for the metric g_L .

Proposition 3.1. *We have*

$$\begin{aligned} D_{\mathbf{e}_1} \mathbf{e}_2 &= -D_{\mathbf{e}_2} \mathbf{e}_1 = \frac{1}{2} \mathbf{e}_3, & D_{\mathbf{e}_1} \mathbf{e}_3 &= D_{\mathbf{e}_3} \mathbf{e}_1 = \frac{1}{2} \mathbf{e}_2, \\ D_{\mathbf{e}_2} \mathbf{e}_3 &= D_{\mathbf{e}_3} \mathbf{e}_2 = -\frac{1}{2} \mathbf{e}_1, & D_{\mathbf{e}_i} \mathbf{e}_i &= 0, \quad i = 1, 2, 3. \end{aligned}$$

[Proof] It is known that the Koszul formula

$$\begin{aligned} 2\langle \nabla_V W, X \rangle &= V\langle W, X \rangle + W\langle X, V \rangle - X\langle V, W \rangle \\ &\quad - \langle V, [W, X] \rangle + \langle W, [X, V] \rangle + \langle X, [V, W] \rangle \end{aligned}$$

holds, see, for instance, [14]. Since

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, \quad [\mathbf{e}_2, \mathbf{e}_3] = [\mathbf{e}_3, \mathbf{e}_1] = 0,$$

one has

$$\begin{aligned} \langle D_{\mathbf{e}_1} \mathbf{e}_2, \mathbf{e}_1 \rangle &= 0, \\ \langle D_{\mathbf{e}_1} \mathbf{e}_2, \mathbf{e}_2 \rangle &= 0, \\ 2\langle D_{\mathbf{e}_1} \mathbf{e}_2, \mathbf{e}_3 \rangle &= \langle \mathbf{e}_3, [\mathbf{e}_1, \mathbf{e}_2] \rangle = \langle \mathbf{e}_3, \mathbf{e}_3 \rangle = -1 \end{aligned}$$

and

$$D_{\mathbf{e}_1} \mathbf{e}_2 = \frac{1}{2} \mathbf{e}_3.$$

Since

$$\begin{aligned} \langle D_{\mathbf{e}_1} \mathbf{e}_3, \mathbf{e}_1 \rangle &= 0, \\ \langle D_{\mathbf{e}_1} \mathbf{e}_3, \mathbf{e}_3 \rangle &= 0, \\ 2\langle D_{\mathbf{e}_1} \mathbf{e}_3, \mathbf{e}_2 \rangle &= \langle \mathbf{e}_3, [\mathbf{e}_2, \mathbf{e}_1] \rangle = \langle \mathbf{e}_3, -\mathbf{e}_3 \rangle = 1, \end{aligned}$$

one has

$$D_{\mathbf{e}_1} \mathbf{e}_3 = \frac{1}{2} \mathbf{e}_2.$$

One can check the others in the same manner. □

3.2. Lorentzian Exterior Product. For tangent vectors

$$\mathbf{v} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 + a_3 \mathbf{e}_3, \mathbf{w} = b_1 \mathbf{e}_1 + b_2 \mathbf{e}_2 + b_3 \mathbf{e}_3$$

in Nil_1^3 , the Lorentzian exterior product $\mathbf{v} \times_L \mathbf{w}$ is computed as

$$\begin{aligned} \mathbf{v} \times_L \mathbf{w} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & -\mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &= (a_2 b_3 - a_3 b_2) \mathbf{e}_1 + (a_3 b_1 - a_1 b_3) \mathbf{e}_2 + (a_2 b_1 - a_1 b_2) \mathbf{e}_3 \end{aligned}$$

which is orthogonal to both \mathbf{v} and \mathbf{w} . One can easily see that $\mathbf{v} \times_L \mathbf{w} = \mathbf{0}$ if and only if \mathbf{v} and \mathbf{w} are linearly dependent.

3.3. Zero mean curvature equation. Let Σ be a graph of a function $z = f(x, y)$ in \mathbb{H}_3 and consider the parametrization $\mathbf{r}(x, y) = (x, y, f(x, y))$ of Σ . Set

$$p = f_x + \frac{y}{2}, \quad q = f_y - \frac{x}{2}.$$

If Σ is minimal, that is, the mean curvature is zero in Nil^3 , the function f satisfies the minimal surface equation

$$(1 + q^2)f_{xx} - 2pqf_{xy} + (1 + p^2)f_{yy} = 0.$$

For the derivation of this equation, see for example [9].

In this section, we are to derive an equation for the mean curvature of the graph Σ to be zero with respect to the Lorentzian metric g_L . First, let us recall some definitions. A point $z \in \Sigma$ is called *spacelike* if the induced metric on $T_z \Sigma$ is Riemannian, *timelike* if the induced metric is Lorentzian and *lightlike* if the induced metric has rank 1. We are to derive the equation when Σ is spacelike, that is, every point of Σ is a spacelike point. The case when Σ is timelike is almost identical. Note that when $z \in \Sigma$ is lightlike, one cannot define the mean curvature.

Now let Σ be a spacelike graph of a function $z = f(x, y)$. Note first that $p^2 + q^2 < 1$ since the graph is spacelike. We now compute the first fundamental form I and the second fundamental form II of Σ . Since

$$\begin{aligned} \mathbf{r}_x &= (1, 0, f_x) = \mathbf{e}_1 + p\mathbf{e}_3, & \mathbf{r}_y &= (0, 1, f_y) = \mathbf{e}_2 + q\mathbf{e}_3, \\ \langle \mathbf{r}_x, \mathbf{r}_x \rangle &= 1 - p^2, & \langle \mathbf{r}_x, \mathbf{r}_y \rangle &= -pq, & \langle \mathbf{r}_y, \mathbf{r}_y \rangle &= 1 - q^2. \end{aligned}$$

one has

$$E = \langle \mathbf{r}_x, \mathbf{r}_x \rangle = 1 - p^2, \quad F = \langle \mathbf{r}_x, \mathbf{r}_y \rangle = -pq, \quad G = \langle \mathbf{r}_y, \mathbf{r}_y \rangle = 1 - q^2.$$

Since

$$\mathbf{r}_x \times_L \mathbf{r}_y = -p\mathbf{e}_1 - q\mathbf{e}_2 - \mathbf{e}_3$$

the unit normal vector field \mathbf{n} to the graph is

$$\mathbf{n} = \frac{1}{W} (-p\mathbf{e}_1 - q\mathbf{e}_2 - \mathbf{e}_3), \quad W = \sqrt{1 - (p^2 + q^2)}.$$

Since the directional derivatives of p and q , $\mathbf{e}_i(p), \mathbf{e}_i(q)$ are computed as

$$\begin{aligned}\mathbf{e}_1(p) &= \left(\frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z} \right) \left(f_x + \frac{y}{2} \right) = f_{xx}, \\ \mathbf{e}_1(q) &= \left(\frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z} \right) \left(f_y - \frac{x}{2} \right) = f_{xy} - \frac{1}{2}, \\ \mathbf{e}_2(p) &= \left(\frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z} \right) \left(f_x + \frac{y}{2} \right) = f_{xy} + \frac{1}{2}, \\ \mathbf{e}_2(q) &= \left(\frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z} \right) \left(f_y - \frac{x}{2} \right) = f_{yy},\end{aligned}$$

one has

$$\begin{aligned}D_{\mathbf{r}_x} \mathbf{r}_x &= D_{(\mathbf{e}_1 + p\mathbf{e}_3)}(\mathbf{e}_1 + p\mathbf{e}_3) \\ &= p\mathbf{e}_2 + f_{xx}\mathbf{e}_3, \\ D_{\mathbf{r}_y} \mathbf{r}_x &= -\frac{p}{2}\mathbf{e}_1 + \frac{q}{2}\mathbf{e}_2 + f_{xy}\mathbf{e}_3, \\ D_{\mathbf{r}_y} \mathbf{r}_y &= -q\mathbf{e}_1 + f_{yy}\mathbf{e}_3.\end{aligned}$$

Then one has the following coefficients of the second fundamental form II.

$$\begin{aligned}l &= \langle D_{\mathbf{r}_x} \mathbf{r}_x, \mathbf{n} \rangle = \frac{1}{W} (-pq + f_{xx}), \\ m &= \langle D_{\mathbf{r}_y} \mathbf{r}_x, \mathbf{n} \rangle = \frac{1}{W} \left(\frac{p^2}{2} - \frac{q^2}{2} + f_{xy} \right), \\ n &= \langle D_{\mathbf{r}_y} \mathbf{r}_y, \mathbf{n} \rangle = \frac{1}{W} (pq + f_{yy}).\end{aligned}$$

Now the mean curvature H of the spacelike graph Σ is computed as

$$H = \frac{1}{2} \frac{lG - 2mF + nE}{EG - F^2}.$$

Then, since

$$\begin{aligned}lG - 2mF + nE &= \frac{1}{W} \left[(-pq + f_{xx})(1 - q^2) + \left(\frac{p^2 - q^2}{2} + f_{xy} \right) pq + (pq + f_{yy})(1 - p^2) \right] \\ &= \frac{1}{W} [(1 - q^2)f_{xx} + 2pqf_{xy} + (1 - p^2)f_{yy}],\end{aligned}$$

one can see that the mean curvature of the graph $z = f(x, y)$ of a function $f(x, y)$ is zero if and only if

$$(1 - q^2)f_{xx} + 2pqf_{xy} + (1 - p^2)f_{yy} = 0.$$

When the graph Σ is timelike, one has the same equation.

3.4. Zero mean curvature surface. We prove the following theorem:

Theorem 3.2. *Let Σ be a surface in \mathbb{H}_3 . If the mean curvature of Σ is zero with respect to both metrics g and g_L , then up to the isometries in Nil^3 , Σ is a part of the horizontal plane $z = 0$, the vertical plane $y = 0$, a helicoid $\tan(\lambda z) = \frac{y}{x}, \lambda \neq 0$ or a hyperbolic paraboloid $z = -\frac{xy}{2}$.*

[Proof] Suppose first that Σ has a point around which can be represented as a graph of a function of (x, y) , say, $z = f(x, y)$. Consider the vector field

$$X = -q\mathbf{e}_1 + p\mathbf{e}_2.$$

Since

$$X = -q\mathbf{e}_1 + p\mathbf{e}_2 = -q\mathbf{r}_x + p\mathbf{r}_y$$

it is tangent to Σ . Since the vector

$$N = \mathbf{r}_x \times \mathbf{r}_y = -p\mathbf{e}_1 - q\mathbf{e}_2 - \mathbf{e}_3$$

is orthogonal to Σ and since $N \times \mathbf{e}_3 = -q\mathbf{e}_1 + p\mathbf{e}_2 = X$, X is orthogonal to both N and \mathbf{e}_3 . Then one has

$$\nabla_X X = \left(q \left(f_{xy} - \frac{1}{2} \right) - pf_{yy} \right) \mathbf{e}_1 + \left(p \left(f_{xy} + \frac{1}{2} \right) - qf_{xx} \right) \mathbf{e}_2.$$

Now, since the mean curvature of $\Sigma \subset \mathbb{H}_3$ is zero with respect to both g and g_L , one has

$$(5) \quad (1 + q^2)f_{xx} - 2pqf_{xy} + (1 + p^2)f_{yy} = 0,$$

$$(6) \quad (1 - q^2)f_{xx} + 2pqf_{xy} + (1 - p^2)f_{yy} = 0.$$

Subtracting two equations, one has

$$(7) \quad q^2 f_{xx} - 2pqf_{xy} + p^2 f_{yy} = 0$$

and then one has finally by (7)

$$\begin{aligned} X \times \nabla_X X &= (-q\mathbf{e}_1 + p\mathbf{e}_2) \times \left[\left(q \left(f_{xy} - \frac{1}{2} \right) - pf_{yy} \right) \mathbf{e}_1 + \left(p \left(f_{xy} + \frac{1}{2} \right) - qf_{xx} \right) \mathbf{e}_2 \right] \\ &= (q^2 f_{xx} - 2pqf_{xy} + p^2 f_{yy}) \mathbf{e}_3 \\ &= 0. \end{aligned}$$

Now, since X and $\nabla_X X$ are of the same direction, the integral curve of X passing through a point in Σ is a geodesic and since X is orthogonal to \mathbf{e}_3 , the geodesic is orthogonal to the fibre. Hence the surface Σ is a horizontally ruled minimal surface in Nil^3 .

If the surface Σ has no point around which Σ is represented as the graph of $f(x, y)$, then it is a vertical cylinder over a curve in the xy plane and has a parametrization $X(s, t) = (x(s), y(s), t)$, $x(0) = y(0) = 0$. By repeating the arguments in Theorem 2.3, one can show that the surface is isometric to the vertical plane $y = 0$. Now this completes the proof. \square

3.5. Remark. If we add (5) and (6), we have

$$f_{xx} + f_{yy} = 0$$

that is, if a graph of a function $z = f(x, y)$ in \mathbb{H}_3 satisfies the condition of Theorem 3.2, f must be a harmonic function. This fact is true for the three dimensional Lorentzian space \mathbb{L}^3 and is the motivation of [11]. We think it is a nontrivial fact and would like to find applications of this fact in the future study.

REFERENCES

- [1] Albuje, A. L., Alías, L. J., Calabi-Bernstein result for maximal surfaces in Lorentz product space, arXiv:0709.4363v1.
- [2] Alías, L. J., Palmer, B., A duality results between the minimal surface equation and the maximal surface equation, *An. Acad. Bras. Ciénc.* **73** (2001), 161–164.
- [3] Bekkar, M., Bouziani, F., Boukhatem, Y., Inoguchi, J., Helicoids and axially symmetric minimal surfaces in 3-dimensional homogeneous spaces, *Differ. Geom. Dyn. Syst.* **9** (2007) 21–39.
- [4] Bekkar, M., Sari, T., Surfaces minimales réglées dans l’espace de Heisenberg \mathbb{H}^3 , *Rend. Sem. Mat. Pol. Torino* **50** (1992), 243–254.
- [5] Daniel, B., The Gauss map of minimal surfaces in the Heisenberg group, arXiv:math/0606299v1.
- [6] Gromov, M., Metric structures for Riemannian and non-Riemannian spaces, Birkhäuser, Boston, 1999.
- [7] Inoguchi, J., Flat translation invariant surfaces in the 3-dimensional Heisenberg group, *J. Geom.* **82** (2005) 83–90.
- [8] Inoguchi, J., Minimal surfaces in the 3-dimensional Heisenberg group, *Differ. Geom. Dyn. Syst.* **10** (2008), 163–169.
- [9] Inoguchi, J., Kumamoto, T., Ohsugi, N., Suyama, Y., Differential geometry of curves and surfaces in 3-dimensional homogeneous spaces I and II, *Fukuoka Univ. Sci. Rep.* **29** (1999), 555–582. **30** (2000), 14–47.
- [10] Kim, Y. W., Koh, S.-E., Shin, H., Yang, S.-D., Helicoids in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$, to appear in *Pacific Jour. Math.*
- [11] Kim, Y. W., Lee, H. Y., Yang, S.-D., Minimal harmonic graphs and their Lorentzian cousin, *Jour. Math. Anal. Appl.* **353** (2009), 666–670.
- [12] Kobayashi, O., Maximal surfaces in the 3-dimensional Minkowski space L^3 , *Tokyo J. Math.* **6** (1983), 297–309.
- [13] Mercury, F., Montaldo, S., Piu, P., A Weierstrass representation formula for minimal surfaces in \mathbb{H}_3 and $\mathbb{H}^2 \times \mathbb{R}$, *Acta Math. Sin. Eng. Ser.* **22** (2005), 1803–1612.
- [14] O’Neill, B., Semi-Riemannian Geometry, Academic Press, San Diego, 1983.
- [15] Rahmani, S., Métriques de Lorentz sur les groupes de Lie unimodulaires de dimension 3, *J. Geom. Phys.* **9** (1992), 295–302.
- [16] Rahmani, N., Rahmani, S., Lorentzian geometry of the Heisenberg group, *Geom. Dedicata* **118** (2006), 133–140.
- [17] Sanini, A., Gauss map of a surface of the Heisenberg group, *Boll. Un. Mat. Ital. B* **11** (2, supp.) (1997), 79–93.

DEPT. OF MATHEMATICS, KOREA UNIVERSITY, SEOUL, KOREA 136-701
E-mail address: ywkim@korea.ac.kr

DEPT. OF MATHEMATICS, KONKUK UNIVERSITY, SEOUL, 143-701, KOREA,
E-mail address: sekoh@konkuk.ac.kr

DEPT. OF MATHEMATICS, KOREA UNIVERSITY, SEOUL, 143-701, KOREA,
E-mail address: distgeo@korea.ac.kr

DEPT. OF MATHEMATICS, CHUNG-ANG UNIVERSITY, SEOUL, 156-756, KOREA,
E-mail address: hshin@cau.ac.kr

DEPT. OF MATHEMATICS, KOREA UNIVERSITY, SEOUL, KOREA 136-701
E-mail address: sdyang@korea.ac.kr